Homework 6 (Due 3/5/2014)

Math 622

February 26, 2014

1. Let $dS(t) = rS(t)\,dt + \sigma(S(t))S(t)\,d\widetilde{W}(t)$ be a local-volatility, risk-neutral price model. We assume that $\sigma(S(t)) > 0$ always. Define $Y(t) = \max_{[0,t]} S(u)$ and $L(t) = \min_{[0,t]} S(u)$. It can be shown that $(S(t), Y(t), L(t))$ is a Markov process.

In this problem we are interested in options with payoff $G(S(T), Y(T), L(T))$ at expiry T. Let $U(t)$ be the price of the option at time t.

a) Show that $U(t)$ can be written in the form $U(t) = u(t, S(t), Y(t), L(t))$ for some function $u(t, x, z, y)$. It is not necessary to represent u in any special way; just argue it exists.

b) We know that $Z(t)$, $t \geq 0$, is a singular process. The same is true of $L(t)$, $t \geq 0$. It is decreasing, $\int_0^t g(s)\,dL(s) = \int_0^t g(s)\mathbf{1}_{\{S(s)=L(s)\}}\,dL(s)$ and the set $\{t;\, S(t) = L(t)\}$ has zero length.

Assuming u has sufficiently many smooth derivatives, derive a p.d.e. with boundary conditions for u. Your solution should specify precisely the domain of (t, x, z, y) space in which the equation holds, and also all relevant terminal and boundary conditions.

2. In section 7.4.3, Shreve shows how to reduce the dimension of the p.d.e. for the floating strike lookback option, by using the scaling $v(t, \lambda x, \lambda y) = \lambda v(t, x, y)$ to write $v(t, x, y) = yu(t, x/y)$ where $u(t, z) = v(t, z, 1)$, and then finding a p.d.e. with boundary conditions for u.

We could also use the scaling to write $v(t, x, y) = xc(t, y/x)$, where $c(t, z) = v(t, 1, y/x)$. If possible, derive a p.d.e. for c with boundary conditions.

3. Assume that S satisfies the risk-neutral, Black-Scholes model,

$$dS(t) = rS(t)\,dt + \sigma S(t)\,\mathrm{d}\widetilde{W}(t).$$

This problem is about analyzing the the average strike call, which is defined on page 1 of the *Lecture 6b notes* to be that option with payoff $(S(T) - S_{\text{ave}}(T))^{+}$.

Let $Y(t) = \int_0^t S(u)\, du$.

a) Show that the price of the can be written as $V(t) = v(t, S(t), Y(t))$ and give an expression for $v(t, x, y)$ of the form $v(t, x, y) = \tilde{E}[L(t, x, y)]$ where L is a random variable, and give an explicit formula for $L(t, x, y)$ in terms of t, x, y and the process $\{W(u) - W(t);\, u \geq t\}$.

b) Find a p.d.e. for v. Be sure to specify the domain of (t, x, y)-space where the equation holds. Derive or write down all boundary and terminal conditions. You will have to specify a boundary condition that fixes the value of v, either as $y \to \infty$ or $y \to -\infty$.

c) Show there is a scaling like that for the floating strike lookback and use it to find a p.d.e. of reduced dimension for characterizing the price of the average strike call.

4. Let

$$
\begin{aligned}
dS(t) &= rS(t)\, dt + \sigma(S(t))S(t)\, d\widetilde{W}(t), S(0) = s_0 && (1)\\
Y(t) &= y_0 + \int_0^T S(u)\, du && (2)\\
Z(t) &= \max_{0 \leq u \leq t} S(u) && (3)
\end{aligned}
$$

Here $s_0 > 0$ is any starting price and y_0 could be any real value.

It can be shown that the vector-valued process $(S(t), Y(t), Z(t))$, $t \geq 0$, is a Markov process.

Consider the option with payoff at T equal to $Z(T) - \frac{1}{T}Y(T)$. Let $V(t)$ be its price at time t.

a) Show there is a function v such that $V(t) = v(t, S(t), Y(t), Z(t))$. It is not necessary to represent v explicitly, just explain why such a function exists.

b) Find a partial differential equation for $v(t, x, y, z)$. State the domain (the region of (t, x, y, z) space) where this equation holds. The method you use to find this equation should also lead to a boundary condition. State this boundary condition and how you derive it. Finally, state a terminal condition on v.